Energy Level Dependent Regular and Non-Regular Behaviours of A Double Pendulum System with Lagrangian Formalism

Lawal J., Ahmad H. and Hassan I.
Department of Mathematics and Statistic, Nuhu Bamalli Polytechnic, Zaria
Corresponding Author: Email: lawaljaafaru@gmail.com; Phone: 08034300238

Abstract
Regular and Non-regular behaviours of a double pendulum system are investigated through the energy level with Lagrangian formalism. The kinetic and potential energies in terms of the generalized coordinates $\{\theta_1,\theta_2\}$ *and their corresponding velocities* $\{\dot{\theta}_1,\dot{\theta}_2\}$ *are obtained. The Lagrangian formalism is used to obtain the energy equations for the double pendulum system. From those equations, we obtain the regular and non-regular behaviours of the double pendulum system. At the energy level* $[1.0301\ J\ ]$*the system's behaviour is cyclic which indicates periodicity, confirming that it is the indication of regular behaviour. Whereas at the energy level* $[1.5779\ J\ ]$ *the system shows random behaviour which is an indication of a non-regular behaviour. As the energy increases to* $[3.0910\ J\ ]$ *the behaviour shows an almost periodic motion which is an indication of a regular behaviour. It is established that at low and high energy values the behaviour of the double pendulum system is regular (non-chaotic) while at the intermediate energy values the motion of the double pendulum system is non-regular (chaotic).Itwas recommended that middle energy values should be given proper consideration when it comes to real-life applications.*

Keywords: Double Pendulum, Energy Level, Regular Behaviour, Non-Regular Behaviour, andLagrangian formalism

INTRODUCTION

A pendulum is a weight suspended from a pivot so that it can swing freely (Gupta *et.al,* 2014). The pendulum system behaves differently at different energy levels, and it is used in many scientific instruments such as seismometers, gravimeters, and accelerometers. However, the energy of the system determines how slow or fast a system is, since energy is the ability to do work (Ambrose and Amos, 2012). The double pendulum system consisting of two-point masses attached by massless rods and free to rotate in a plane is one of the simplest dynamical systems to exhibit non-regular (chaos) behaviour but with small (low) energy or extremely large energy, the behaviour of the system changes. When a pendulum bob is displaced from its equilibrium position (where it is not moving), work is done on the pendulum bob and so it acquires gravitational potential energy (Herbert, 1980). This is the energy an object has because of its position within the Earth's gravitational field. When a pendulum bob has gravitational potential energy, it is subject to a restoring force due to gravity, a force that will accelerate it back towards the equilibrium position. When the pendulum bob is released, the restoring force combined with the bob's mass causes it to oscillate about the equilibrium position, swinging back and forth. As the pendulum bob begins to move, its gravitational potential energy is transformed into kinetic energy, or energy of motion (Jewett, 2004). When the pendulum is at the part of its period where it is at a 90 degrees angle with the horizontal, its original gravitational potential energy has been entirely transformed into kinetic energy. As it continues to swing and move upward on the arc, the kinetic energy is transformed back to gravitational potential energy. This cycle repeats itself until the pendulum stops swinging (Baker and Blackburn,

2005). When a pendulum oscillates there are always some points in its oscillatory motion where it is momentarily at rest and other points at which it is moving with maximum speed. Consequently, the kinetic energy of the system is sometimes zero and sometimes a maximum. However, the potential energy is maximum when the kinetic energy is minimum and vice versa. The transformation of potential energy into kinetic energy is depicted in figure 1.1

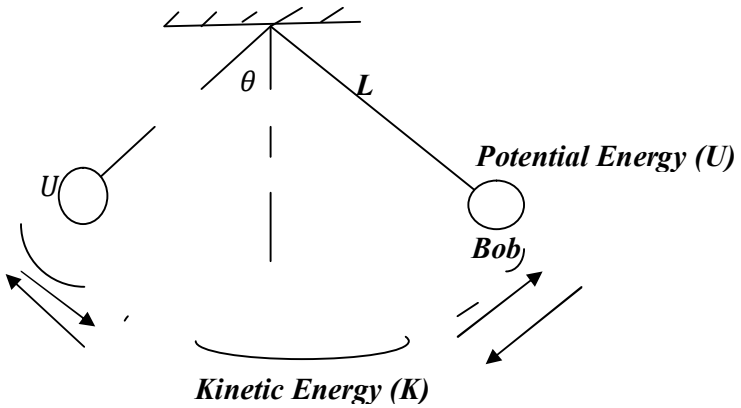

Figure 1.1: Transformation of potential energy into kinetic energy

A Double pendulum suspended from a fixed point. A double pendulum is a pendulum attached to the end of another pendulum. Thus, the axes at the upper end of each rod allow for frictionless rotation in the plane. The upper mass is m_1 and the lower is m_2. The lengths of the rods are l_1 (upper) and l_2 (lower). The motion will be described in terms of the angles to the vertical line θ_1 for the upper and θ_2 for the lower, as shown below.

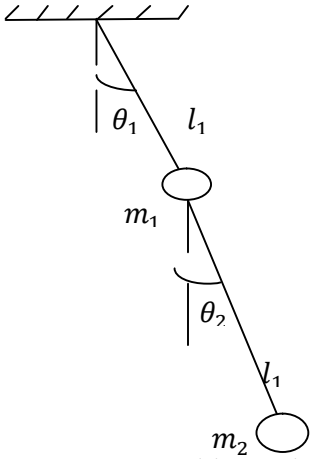

Figure1.2: Double pendulum

Recently several researchers have studied the dynamics of the double pendulum system. Amongst them is Gupta *et.al*(2014) who investigated the dynamics

of the double pendulum system. The chaotic behaviour of the double pendulum system is studied numerically using Hamilton's equations. By varying the mass and length of the double pendulum system, it was observed that the chaotic tendency of the double pendulum increases with mass and length.

Biglari and Jami (2016) carried out a numerical analysis on a system of the double pendulum with a Hamiltonian equation of motion, this equation was used to obtain the energy equations of motion of the double pendulum. In their work, the behaviour of the system was analyzed through the variations of the system's energy and initial conditions. They found that the system's behaviour changes as the energy increases from low, intermediate to high energies. In addition, it was seen that at low energy the motion of the system is periodic, at intermediate energy the motion is chaotic, and at high energy, the motion is quasiperiodic. The aforementioned researchers used the Hamiltonian approach, which consists of angular displacements, angular velocities, and generalized momenta, ignoring the angular acceleration. Angular acceleration is the rate of change of angular velocity with respect to time. In actual fact, when a double pendulum system is in motion, there will be angular displacements, angular velocities and with little change, angular acceleration will be in the picture. It is noted that to tackle the behaviour of the double pendulum system with corresponding angular acceleration, we employ Euler Lagrange's approach, which captures the angular displacements, angular velocities, and as well as angular accelerations. In this research, we are particularly concerned with the behaviour the system of double pendulum exhibits. Thus, we first obtain the kinetic and potential energies of the upper pendulum and then kinetic and potential energies of the combined pendulums (double pendulum). The Lagrangian formalism is used to obtain the energy equations for the double pendulum system. From these equations, we obtained the regular and non-regular behaviours of the double pendulum system.

AIM AND OBJECTIVES

The aim of this research is to formulate the energy equations of the double pendulum system and use the equations to identify the regular and non-regular behaviour of the double pendulum system at different energy level. This shall be achieved under the following objectives;

(i) To formulate the energy equations of motion using Euler Lagrange's formalism

(ii) To obtain the solutions to the energy equations in (i) analytically

(iii)To indentify the regular and non-regular behaviour of the double pendulum system at different energy level.

Definitions

Energy level is a state of energy at which the behaviour of the double pendulum system can be predicted or not predicted

Regular Behaviour: A motion of double pendulum is regular if the baheviour of the double pendulum system can determined or predicted

Non-regular Behaviour: A motion of double pendulum system is non-regular if the baheviour of the double pendulum system cannot be determined or predicted

Generalized coordinates is the set of independent coordinates explicitly defining position of the system, number of generalized coordinates is equal to the number of degrees of freedom of the system (Raymond, 1983).

METHODOLOGY

Energy Equations of the Double Pendulum System Using Euler Lagrange's Equation

In this section, we will obtain the desired energy equations of the double pendulum system using Euler Lagrange's equation.

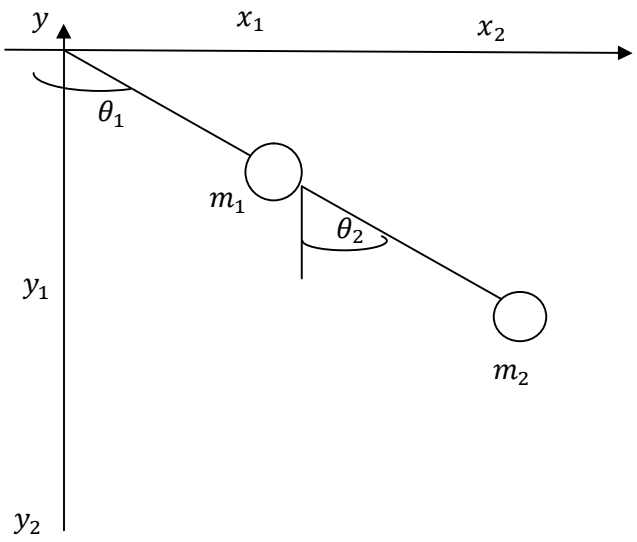

Figure 2.1:Positions of pendulum bobs m_1 and m_2

For the two masses m_1 and m_2 their positions are:

For$m_1 x_1 = l_1 \sin\theta_1$ (2.1)

$y_1 = l_1 \cos\theta_1$ (2.2)

And for m_2

$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$ (2.3)

$y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$ (2.4)

Where (x_1,y_1) and (x_2,y_2) represent the coordinates of the pendulum bob in a plane,l_1 and l_2 are the lengths of the rod, and θ_1 and θ_2 are the displaced angles.

The velocities are:

$\dot{x_1} = l_1 \cos(\theta_1)\,\dot{\theta}_1$ (2.5)

$\dot{y_1} = -l_1 \sin(\theta_1)\,\dot{\theta}_1$ (2.6)

and

$\dot{x_2} = l_1 \cos(\theta_1)\,\dot{\theta}_1 + l_2 \cos(\theta_2)\,\dot{\theta}_2$ (2.7)

$\dot{y_2} = -l_1 \sin(\theta_1)\,\dot{\theta}_1 - l_2 \sin(\theta_2)\,\dot{\theta}_2$ (2.8)

The kinetic and potential energies are derived as:

$K_T = \frac{1}{2} m_1 {l_1}^2 {\dot{\theta}_1}^2 + \frac{1}{2} m_2 \left[{l_1}^2 {\dot{\theta}_1}^2 + {l_2}^2 {\dot{\theta}_2}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$ (2.9)

and,

$U = -m_1 g l_1 \cos(\theta_1) - m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$ (2.10)

respectively.

Then, the Lagrangian $L = K_T - U$of the generalized coordinates (θ_1, θ_2) is:

$L = (m_1 + m_2)\frac{1}{2} {l_1}^2 {\dot{\theta}_1}^2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + m_1 g l_1 \cos(\theta_1) + m_2 g (l_1 \cos(\theta_1) + l_2 \cos(\theta_2))$ (2.11)

And the Euler Lagrange's equation which is given by:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0, \quad i = 1, 2, \ldots\ldots$$ (2.12)

Equation (2.12) will be used to obtain the desired energy equations as:

$$\ddot{\theta}_1 = -\left[\frac{m_2 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2)}{(m_1+m_2)l_1} + \frac{m_2 l_2 \dot{\theta}_2{}^2 \sin(\theta_1-\theta_2)}{(m_1+m_2)l_1} + \frac{g\sin(\theta_1)}{l_1}\right] \quad (2.13)$$

And

$$\ddot{\theta}_2 = \frac{1}{l_2}\left[l_1\dot{\theta}_1{}^2 \sin(\theta_1-\theta_2) - l_1\ddot{\theta}_1\cos(\theta_1-\theta_2) - g\sin(\theta_2)\right] \quad (2.14)$$

Simplifying equation (2.13), we obtain

$$\ddot{\theta}_1 = \frac{m_2 g \sin(\theta_2)\cos(\theta_1-\theta_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)} - \frac{(m_1+m_2)g\sin(\theta_1)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)} - \frac{m_2 l_2 \dot{\theta}_2{}^2 \sin(\theta_1-\theta_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)} - \frac{m_2 l_1 \dot{\theta}_1{}^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(\theta_1-\theta_2)} \quad (2.15)$$

Simplifying equation (2.14), we arrived at

$$\ddot{\theta}_2 = \frac{(m_1+m_2)l_1\dot{\theta}_1{}^2 \sin(\theta_1-\theta_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(\theta_1-\theta_2)} + \frac{m_2 l_2 \dot{\theta}_2{}^2 \sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(\theta_1-\theta_2)} + \frac{(m_1+m_2)g\sin(\theta_1)\cos(\theta_1-\theta_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(\theta_1-\theta_2)} - \frac{(m_1+m_2)g\sin(\theta_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(\theta_1-\theta_2)} \quad (2.16)$$

Equations (2.15) and (2.16) are the desired energy equations of the double pendulum system.

In order to solve the energy equations, we convert equations (2.15) and (2.16) respectively, to a system of first-order ordinary differential equations by the following transformation:

$$p_1 = \theta_1 \quad (2.17)$$

$$p_2 = \theta_2 \quad (2.18)$$

$$p_3 = \dot{\theta}_1 \quad (2.19)$$

And

$$p_4 = \dot{\theta}_2 \quad (2.20)$$

with these, we obtain the system of first-order ordinary differential equations

$$\dot{p}_3 = \frac{m_2 g \sin(p_2)\cos(p_1-p_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(p_1-p_2)} - \frac{(m_1+m_2)g\sin(p_1)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(p_1-p_2)} - \frac{m_2 l_2 p_4{}^2 \sin(p_1-p_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(p_1-p_2)} - \frac{m_2 l_1 p_3{}^2 \sin(p_1-p_2)\cos(p_1-p_2)}{(m_1+m_2)l_1 - m_2 l_1 cos^2(p_1-p_2)} \quad (2.21)$$

And,

$$\dot{p}_4 = \frac{(m_1+m_2)l_1 p_3{}^2 \sin(p_1-p_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(p_1-p_2)} + \frac{m_2 l_2 p_4{}^2 \sin(p_1-p_2)\cos(p_1-p_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(p_1-p_2)} + \frac{(m_1+m_2)g\sin(p_1)\cos(p_1-p_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(p_1-p_2)} - \frac{(m_1+m_2)g\sin(p_2)}{(m_1+m_2)l_2 - m_2 l_2\cos^2(p_1-p_2)} \quad (2.22)$$

Solution to Energy equations

The solutions to the energy equations are obtained using MATLAB, when specific values for the following parameters l_1, l_2, m_1, m_2, p_1, p_2 , p_3 , p_4 are used.

For $l_1 = l_2 = 1$ and $m_1 = m_2 = 1$, we obtained $_1 = 0.0$, $y_1 = -1$ and $x_2 = 0.4794$, $y_2 = -1.8776$

Similarly, for $l_1 = l_2 = 1$, $m_1 = m_2 = 1$, $p_1 = 0.1$, $p_2 = 0.2122$, $p_3 = 0.0$, $p_4 = 0.0$ and $g = 9.8$, we obtained $\dot{p}_3 = 1.0301$ J and $\dot{p}_4 = 1.5779$ J

Behaviour of the Double Pendulum System

Depending on the given initial conditions, used in section 2.2, the double pendulum exhibits regular, non-regular (chaotic) and almost regular behaviour.

DISCUSSION

Plots of Figures 3.1, 3.2 and 3.3: In each graph, the inner and outer bobs showed regular behaviour confirming that it is an indication of a regular behaviour because each pendulum bob maintains its path indefinitely. As we increased the energy to $\dot{p}_3 = 1.8960\ J$ and $\dot{p}_4 = 1.6994\ J$ with initial conditions $[p_1, p_2, p_3, p_4]$ = [5.2, 4, 0.0, 0.0] and $g = 9.8$, we obtained figures 3.4 and 3.5. In this case, the system shows random behaviour which is an indication of a non-regular behaviour. This happens when one of the pendulum bobs or both deviates from its path. However, as

we further increased the energy to $\dot{p}_3 = 9.9554\ J$ and $\dot{p}_4 = 3.0910\ J$ with initial conditions $[p_1,\ p_2,\ p_3,\ p_4]$ = [5, 3.3, 0.1, 0.4] and $g = 9.8$, the plot in figure 3.6 is obtained and the system's behaviour shows almost regular behaviour.

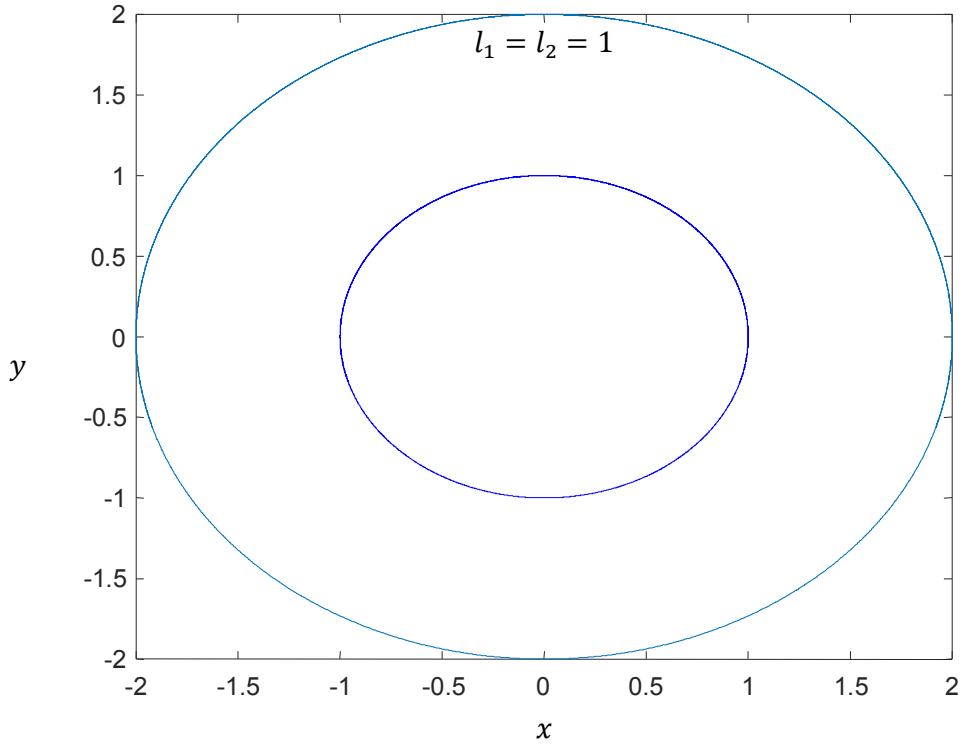

Figure 3.1: Cyclic behaviour (Regular)

For $l_1 = l_2 = 1$, at energy level $[1.0301\,J\,]$ The inner and outer pendulum bobs showed regular behaviour because each pendulum bob maintains its path indefinitely.

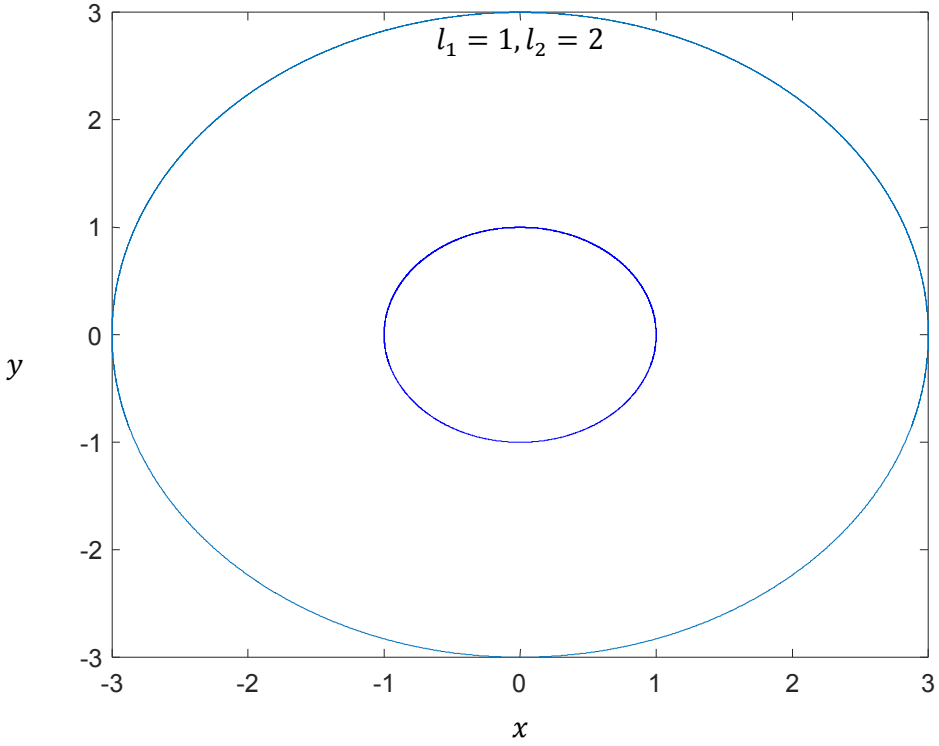

Figure 3.2: Regular

For $l_1 = 1, l_2 = 2$, at energy level $[1.0301\,J\,]$

The inner and outer pendulum bobs showed regular behaviour because each pendulum bob maintains its path indefinitely.

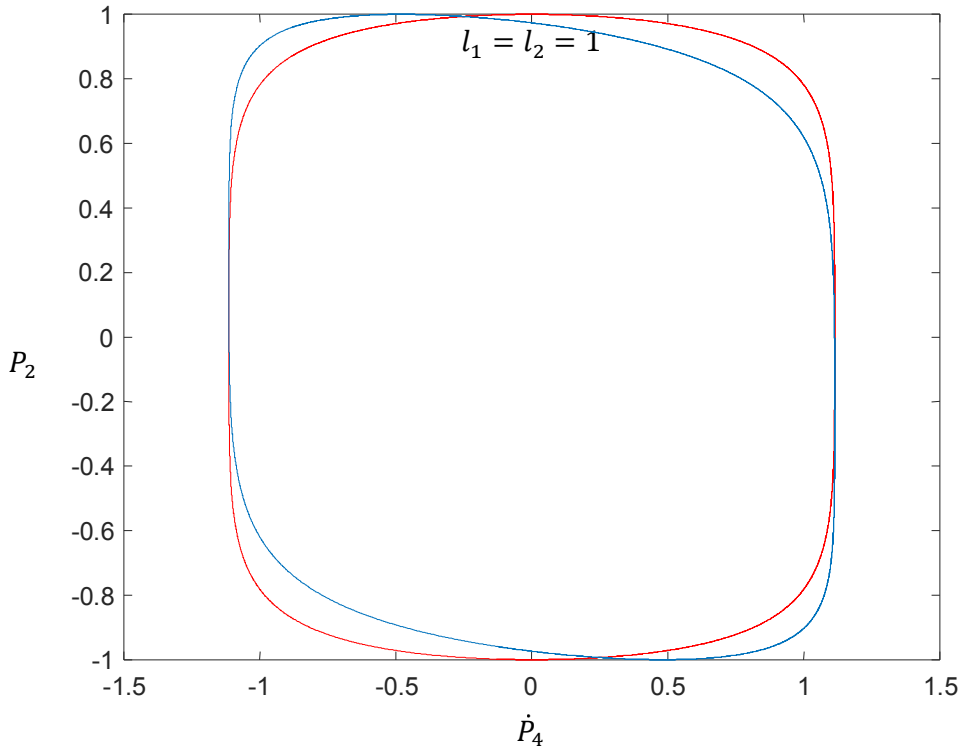

Figure 3.3: Regular

For $l_1 = l_2 = 1$, at energy level $[1.2301\,J\,]$ The inner and outer pendulum bobs showed regular behaviour but with little interaction between the pendulum bobs as a result of energy level changes.

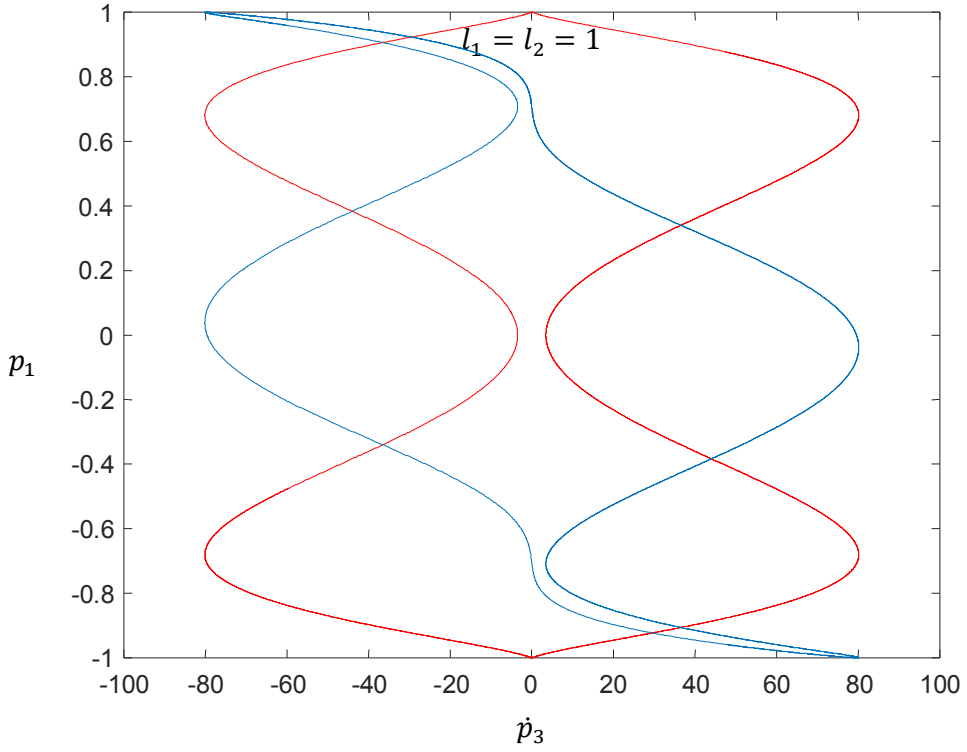

Figure 3.4: Non-regular (Chaotic)

For $l_1 = l_2 = 1$, at energy level $[1.5779\,J\,]$
In this case, the system shows random behaviour which is an indication of a non-regular behaviour. This happens when both of the pendulum bobs deviate from their paths.

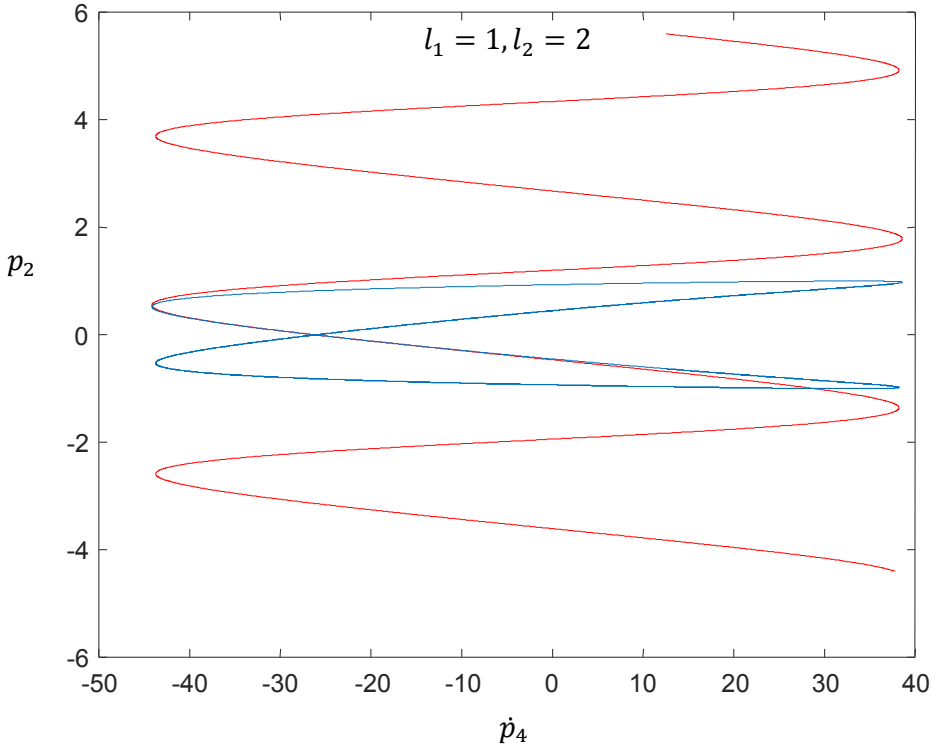

Figure 3.5: Non-regular

For $l_1 = 1,\ l_2 = 2$, at energy level $[1.5779\,J\,]$
The system of double pendulum shows random behaviour which is an indication of a non-regular behaviour. This happens when one of the pendulum bobs deviates from its path.

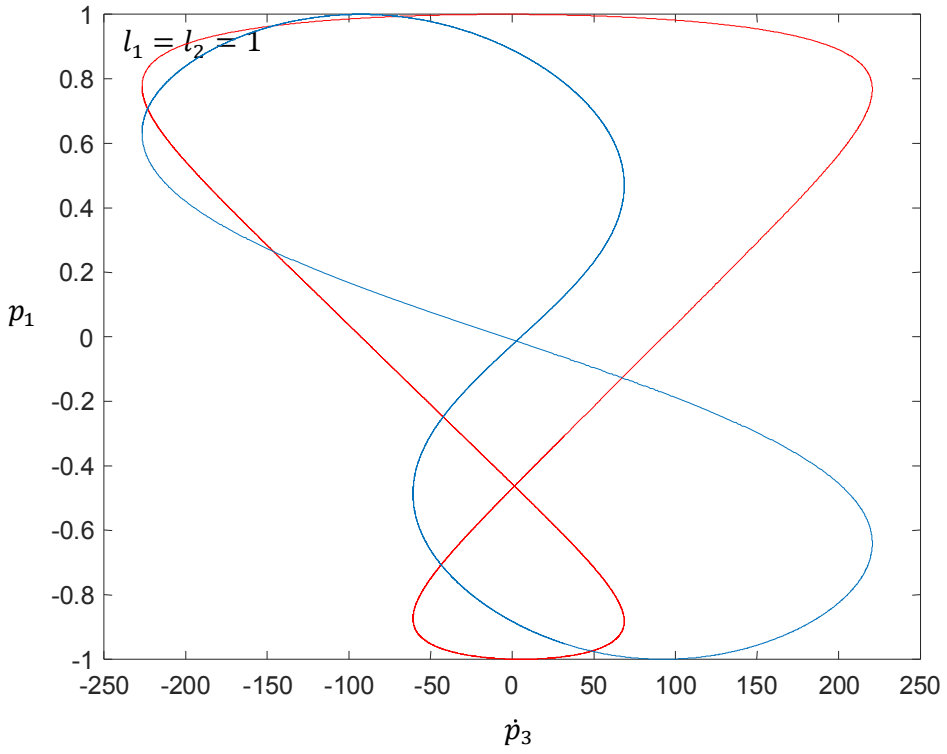

Figure 3.6: Almost regular

For $l_1 = l_2 = 1$, at energy level $[3.0910\ J\ ]$
The inner and outer pendulum bobs showed almost regular behaviour because each pendulum bob possesses extremely large energy, and start regaining their normal path.

CONCLUSION

A double pendulum is a system with great sensitivity to initial conditions. Thus, as the initial condition changes the behaviour of the system changes. However, as the double pendulum begins to move, there is always a tendency to exhibit different behaviour depending on the given initial conditions. This behaviour could be regular, non-regular (chaotic) or almost regular. For the regular behaviour, the double pendulum's motion is repetitive and predictable while the system of double pendulum produces non-regular (chaotic) behaviour when the motion of the system is non-repetitive and unpredictable, and it is almost regular when the behaviour of the system is predictable. This occurs virtually at high levels of energy.

RECOMMENDATIONS

The result of this research shows that the motion of the double pendulum system is regular when the energy level is low or too high and it is non-regular (chaotic) when the energy level is neither low nor too high. This shows that middle energy values should be given proper consideration when it comes to real-life applications.

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